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On the stability of gravity waves on deep water

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This note presents numerical results on the stability of large-amplitude gravity waves on deep water. The results are then used to predict new two-dimensional superharmonic instabilities. They are due to collisions of eigenvalues of opposite signatures, confirming the recent condition for instability of MacKay & Saffman (1986).

1. Introduction

Recently, numerical computations have been used to analyse the stability of water waves. The main advantage is that there exists in principle no restriction concerning the wave steepness value. Early work was limited to two-dimensional instability (Longuet-Higgins 1978*a*) but an extension to three-dimensional perturbations was made by McLean (1982) (see also McLean *et al.* 1981). Besides recovering many of the results of approximate model equations like those developed by Whitham (1974), Benjamin & Feir (1967), Zakharov (1968), Dysthe (1979) and Stiassnie & Shemer (1984), a new fundamental result was obtained: for small wave steepness, the most unstable disturbances are two-dimensional while, for steeper waves, the instabilities are predominantly three-dimensional. Even the strong two-dimensional instabilities found by Longuet-Higgins (1978*b*) (see also Branger, Ramamonjiarisoa & Kharif 1986) beyond $ak = 0.405$ are dominated by the three-dimensional instabilities.

Extending McLean's work, Kharif (1987) showed that the instabilities are no longer predominantly three-dimensional when the wave steepness reaches a value of approximately 0.429.

Very recently, the present authors (Kharif & Ramamonjiarisoa 1988) reported an important result concerning the relative strength of McLean's class I and class II instabilities for basic wave steepness larger than 0.41. Some additional results concerning the latter are presented herein and then attention is focused on superharmonic perturbations. Generally, when unstable these perturbations do not possess the maximum growth rate, but we found cases where they grow at rates very close to that maximum. This may be found of some practical interest since the most unstable subharmonic and superharmonic instabilities are comparable at very high wave steepness.

MacKay & Saffman (1986) gave a necessary condition for instability based on the Hamiltonian nature of the system (Zakharov 1968) and the notion of the signature of an eigenvalue. This condition is confirmed numerically for the instability corresponding to the collision of eigenvalues denoted (iii) by MacKay & Saffman. Then, a new 'bubble' of two-dimensional superharmonic instability is found around

$ak = 0.35$ and another, belonging to class II, is expected to occur at about $ak = 0.433$.

2. Mathematical formulation and numerical schemes

The mathematical formulation of the instability of steady two-dimensional gravity waves on deep water to two- or three-dimensional small perturbations is now well known and is omitted here (for details, see e.g. McLean 1982). It suffices to recall that the analysis, based on Floquet theory, reduces to an eigenvalue problem for the complex frequency of the perturbation, σ.

Let $\eta'(x,y,t)$ be the vertical deflection level of the free surface and $\varphi'(x,y,z,t)$ the velocity potential related to the perturbation. These quantities can be written as

$$\begin{pmatrix}\eta' \\ \varphi'\end{pmatrix} = \mathrm{e}^{-\mathrm{i}\sigma t}\,\mathrm{e}^{\mathrm{i}(px+qy)}\begin{pmatrix}\sum_{-\infty}^{+\infty} a_j\,\mathrm{e}^{\mathrm{i}jx} \\ \sum_{-\infty}^{+\infty} b_j\,\mathrm{e}^{\mathrm{i}jx}\,\mathrm{e}^{[(p+j)^2+q^2]^{\frac{1}{2}}z}\end{pmatrix}, \tag{1}$$

where p is the x-wavenumber and q the y-wavenumber.

Then the eigenvalue problem corresponding to the linear stability problem is

$$\boldsymbol{A}\boldsymbol{u} = \mathrm{i}\sigma\boldsymbol{B}\boldsymbol{u} \tag{2}$$

where $\boldsymbol{u} = (a_j, b_j)$ and the complex matrices $\boldsymbol{A}, \boldsymbol{B}$ depend on the unperturbed wave steepness ak and the wave vector (p,q).

By truncating the series (1) with j in the range $-M < j < M$ the dimension of the matrices $\boldsymbol{A}, \boldsymbol{B}$ is $4M+2$. The integer M is increased until the eigenvalue, σ, has converged. The convergence of the eigenvalues is known to be sensitive to the accuracy of the undisturbed wave. Here, the computation was performed using the iterative scheme developed by Longuet-Higgins (1985) on the basis of a parametric representation of the free surface. In practice, the series are truncated at a finite number of harmonics which is increased when the wave steepness is increased. Solutions are found for values of the basic wave steepness less than about 0.434. At this limiting value the Jacobian matrix is singular and the method fails because the parameter chosen to determine the basic wave is stationary. This limitation may be overcome by choosing a parameter which varies monotonically through the complete range of ak. At a wave amplitude very close to the previous value, 2300 coefficients were necessary to achieve an accuracy of 10^{-10} in the parametric representation, and were computed on a CRAY-2 computer. The corresponding eigenvalues were accurate to three significant figures. The necessity of taking a very large number of harmonics when the wave steepness is large has been pointed out by Kharif & Ramamonjiarisoa (1988, table I).

When the steepness of the undisturbed wave is zero, the eigenvalues are

$$\sigma_n^s = -(p+n)+s[(p+n)^2+q^2]^{\frac{1}{4}}, \quad s = \pm 1.$$

The sign of s determines the direction of propagation of the disturbance relative to the unperturbed wave. The eigenvalues are real for zero ak and, consequently, the perturbations are stable. Instabilities can arise when the parameter $\mu = ak$ increases.

Recently, taking advantage of fundamental work on Hamiltonian systems, MacKay & Saffman (1986) formulated a necessary condition for instability in terms of the collision of two eigenvalues of opposite signatures or at zero frequency. A change from stability to instability can occur only if, for some ak, two modes have the same frequency:

$$\sigma^{s_1}_{n_1}(p,q,ak) = \sigma^{s_2}_{n_2}(p,q,ak).$$

In a fixed frame of reference this relation is equivalent to the resonant conditions

$$\boldsymbol{k}_1 = \boldsymbol{k}_2 + N\boldsymbol{k}_0,$$
$$\omega_1 = \omega_2 + N\omega_0.$$

The vectors $\boldsymbol{k}_i$ are the wavevectors and ω_i the frequencies. The subscripts 0, 1 and 2 refer, respectively, to the undisturbed wave and to the perturbations. The integers $N = 2, 3, 4, 5$ denote, respectively, quartet, quintet, sextet, etc., interactions.

The analysis of two-dimensional superharmonic instabilities on the basis of MacKay & Saffman's necessary condition mentioned previously was done by a different scheme. It consisted in resolving the eigenvalue problem first set by Longuet-Higgins (1978*a*), using the technique due to Tanaka (1983) to reduce the order of the matrices by a factor of two. The eigenvalue problem is

$$-\mathrm{i}\sigma \boldsymbol{A}'\boldsymbol{u}' = \boldsymbol{B}'\boldsymbol{u}',$$

where $\boldsymbol{A}'$ and $\boldsymbol{B}'$ are now real matrices depending on the coefficients in the harmonic representation of the basic wave and $\boldsymbol{u}'$ is the column vector representing the perturbation.

The matrix problem can be cast in the form

$$-\mathrm{i}\sigma\left(\begin{array}{c|c} \boldsymbol{A}'_{11} & 0 \\ \hline 0 & \boldsymbol{A}'_{22}\end{array}\right)\begin{pmatrix}\boldsymbol{u}'_1 \\ \boldsymbol{u}'_2\end{pmatrix} = \left(\begin{array}{c|c} 0 & \boldsymbol{B}'_{12} \\ \hline \boldsymbol{B}'_{21} & 0\end{array}\right)\begin{pmatrix}\boldsymbol{u}'_1 \\ \boldsymbol{u}'_2\end{pmatrix}.$$

Then by eliminating $\boldsymbol{u}'_2$, or $\boldsymbol{u}'_1$, one obtains

$$(-\mathrm{i}\sigma)^2 \boldsymbol{I}\boldsymbol{u}'_1 = (\boldsymbol{A}'^{-1}_{11}\boldsymbol{B}'_{12}\boldsymbol{A}'^{-1}_{22}\boldsymbol{B}'_{21})\boldsymbol{u}'_1,$$
$$(-\mathrm{i}\sigma)^2 \boldsymbol{I}\boldsymbol{u}'_2 = (\boldsymbol{A}'^{-1}_{22}\boldsymbol{B}'_{21}\boldsymbol{A}'^{-1}_{11}\boldsymbol{B}'_{12})\boldsymbol{u}'_2,$$

where is the identity matrix.

The method of resolution briefly described here performs better than the collocation method. Reasons for this were given in some detail by Zhang & Melville (1986) and it will be shown here that reliable results on the coalescence of eigenvalues can be obtained even in the neighbourhood of $ak = 0.426$.

3. Results and discussion

Computations by McLean (1982) limited to wave steepness smaller than 0.41 showed that the maximum growth rate of class II ($m = 1$) instabilities occurred for $p = 0.5$ and $q \neq 0$ while the maximum growth rates of class I ($m = 2$) and class II ($m = 2$) instabilities occurred for, respectively, $p = 0$, $q \neq 0$ and $p = 0.5$, $q \neq 0$. However, these most unstable perturbations are phase locked to the unperturbed wave.

An extension of these computations up to a wave steepness of 0.434 was performed by Kharif & Ramamonjiarisoa (1988) in order to compare the relative strength of class I versus class II instabilities. The extended stability diagram is shown in figure 1. This figure displays instability bands in the (p, q)-plane for class I and class II instabilities with $m = 1$ and $m = 2$ for various wave steepness. The solid lines are the resonance curves from the linear dispersion relation for $ak = 0$. The shaded regions correspond to values of p, q for which instabilities arise. For $m = 2$ the instability

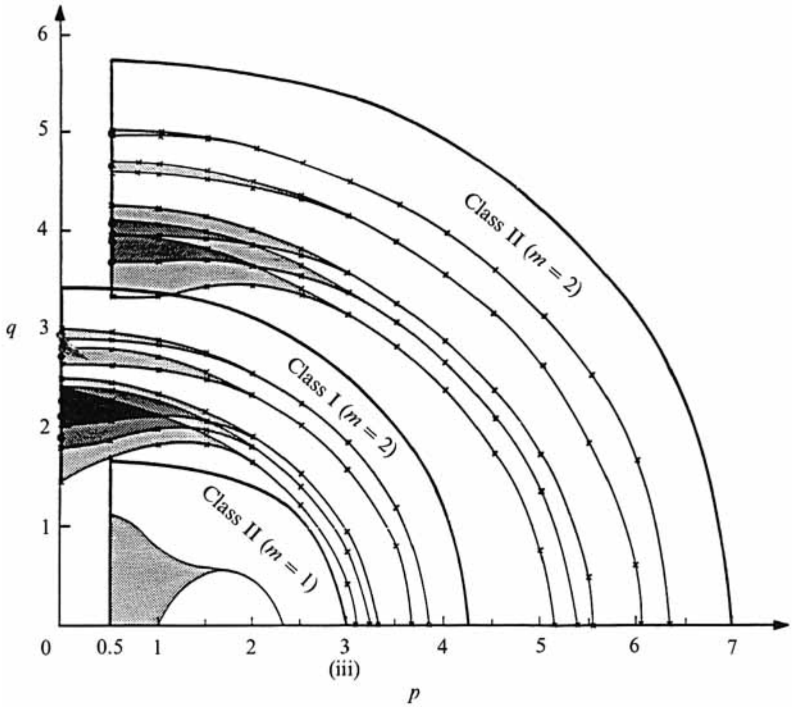

FIGURE 1. Instability regions of class I and class II ($m = 1$ and $m = 2$) for various ak. For $m = 2$, $ak = 0, 0.30, 0.35, 0.40, 0.41, 0.42$ and for $m = 1$, $ak = 0, 0.43$. Solid lines show collisions for $ak = 0$; shaded regions show values of p, q for which the modes are unstable. The dots label the points of maximum growth rates (see Kharif & Ramamonjiarisoa 1988). Point (iii) refers to a collision of eigenvalues identified by MacKay & Saffman (1986).

regions of class I and class II correspond successively to $ak = 0, 0.30, 0.35, 0.40, 0.41, 0.42$. For $m = 1$, only the curve and the band of instability corresponding to, respectively, $ak = 0$ and $ak = 0.43$ are plotted. Superharmonic perturbations are associated with integer values of p; otherwise the perturbations are subharmonic. It is seen that the most unstable modes are subharmonic modes, either two-dimensional or three-dimensional. The theoretical and numerical results are in agreement with observations for small to moderate basic wave steepness (Feir 1967; Su 1982).

The previous results give some insight into the phenomenon of frequency or wavenumber downshifting during the evolution of nonlinear wave trains: at small ($\lesssim 0.30$) wave steepness, Lake *et al.* (1977) found that the cubic Schrödinger equation (Zakharov 1968) did not fully explain the frequency downshift they observed. The recurrent nature of the solutions of such an equation would explain this inadequacy. At moderate and high steepness, the most striking result of the computations is the existence of wavelength doubling associated with the most unstable perturbations ($p = 0.5$). At moderate steepness the wave-field pattern is fully three-dimensional, a fact confirmed by experimental results (Su 1982). Two consecutive structures are phase-shifted in the transverse direction. When the wave steepness increases, the transverse wavenumber q decreases rapidly to zero so that for a basic wave approaching the breaking stage, our computations showed the possibilities of wavelength doubling in a basically two-dimensional motion.

While the Benjamin–Feir side-band instabilities constitute free waves, the McLean class I and class II most unstable modes are bound to the basic wave. These facts have to be kept in mind when trying to utilize these fundamental results to interpret the experimental data.

In the following, we shall focus on superharmonic perturbations. In particular this is motivated by the need to better understand the dynamics of small waves riding over longer waves occurring, for example, in the interaction between microwaves and water surface waves.

For the two-dimensional case, the numerical calculations developed by Longuet-Higgins (1978*a*) showed the perturbation to be neutrally stable up to $ak = 0.42$. Then by extrapolating the numerical results he proposed that instability would appear at the steepness ak where the phase velocity is stationary. An exception was found by Tanaka (1983). Tanaka's instability can now be recovered by three-dimensional computations. McLean (1982) found that at $ak = 0.405$, the stability boundary of class II ($m = 1$) perturbations in the (p, q)-plane touches the p-axis at $p = 0.5$. This yields to the two-dimensional subharmonic instability discovered by Longuet-Higgins (1978*b*). At $ak = 0.41$, the range of two-dimensional instabilities has increased around $p = 0.5$. By extending the McLean computation we found the instability domain to increase further when ak increases and to finally reach the points $(p, q) = (0, 0)$ and $(p, q) = (1, 0)$ at $ak = 0.4292$. This corresponds to Tanaka's instability. The instability is stationary with respect to the unperturbed wave, implying the existence of a trivial bifurcation corresponding to a horizontal translation. For $ak = 0.4292$ there is no further extension of the boundaries of the unstable region along the p-axis; this agrees with previous two-dimensional computations of Longuet-Higgins (1986).

The shaded region of class II ($m = 1$) instabilities plotted in figure 1 corresponds to $ak = 0.4303$ and illustrates the above points. Thus, Tanaka's instability belongs to a McLean class II ($m = 1$) perturbation. This is consistent with the conclusion of Longuet-Higgins (1986) and that of MacKay & Saffman (1986). By extending the computations to three-dimensional cases ($q \neq 0$) we found that two-dimensional perturbations are the most unstable. Additional calculations have been made to compare the growth rates of Tanaka's instabilities ($p = 0, q = 0$) with those of subharmonic instabilities ($p = 0.5$) of class II ($m = 1$). The growth rates were found to match each other at about $ak = 0.438$. In fact within the framework of the MacKay & Saffman necessary condition, recalled in §2, Tanaka's instability corresponds to a collision of eigenvalues at zero frequency. Also, MacKay & Saffman identified examples of instability due to a collision of eigenvalues of opposite signatures. The identification is based on the fact that a superharmonic two-dimensional instability will occur if the instability zones in the (p, q)-plane intersects the p-axis at integer values when the basic wave steepness varies. They presented a complete proof of instability for ak close to 0.24 as the instability region relative to class I ($m = 2$) crosses the p-axis at $p = 4$. The instability is then due to the collision of the modes $p+m = 4+2$ and $p-m = 4-2$ of opposite signatures. Then, they suggested without a complete proof the presence of another instability around $ak = 0.40$ associated with the crossing at $p = 3$ of the class I ($m = 2$) instability zone: the modes $p+m = 3+2$ and $p-m = 3-2$ of opposite signatures would collide around such a wave steepness.

As an example, table 1 gives the convergence and the values of the square of the frequency with respect to the order of truncation N at $ak = 0.42628$ where an instability occurs. Note that very accurate values were obtained with $N = 240$. Figure 2 summarizes the result in terms of a 'bubble of instability' arising within a very narrow range of the wave amplitude. The maximum growth rate is of the order

N	$\mathrm{Re}(\sigma^2)$	$\mathrm{Im}(\sigma^2)$ $(\times 10^2)$
240	3.736004	0.121260
250	3.736830	0.1182383
260	3.736781	0.1159169
270	3.736748	0.1142247
280	3.736726	0.1130301
290	3.736711	0.1122035
300	3.736702	0.1116384
310	3.736695	0.1112586
320	3.736691	0.1109963
330	3.736688	0.1108277
340	3.736686	0.1107041
350	3.736685	0.1106324
360	3.736684	0.1105742
370	3.736683	0.1105442
380	3.736683	0.1105209

TABLE 1. Convergence of the square of the frequency, σ^2, of the superharmonic instability due to the collision of the modes $(n,s) = (5,1)$ and $(n,s) = (1,-1)$

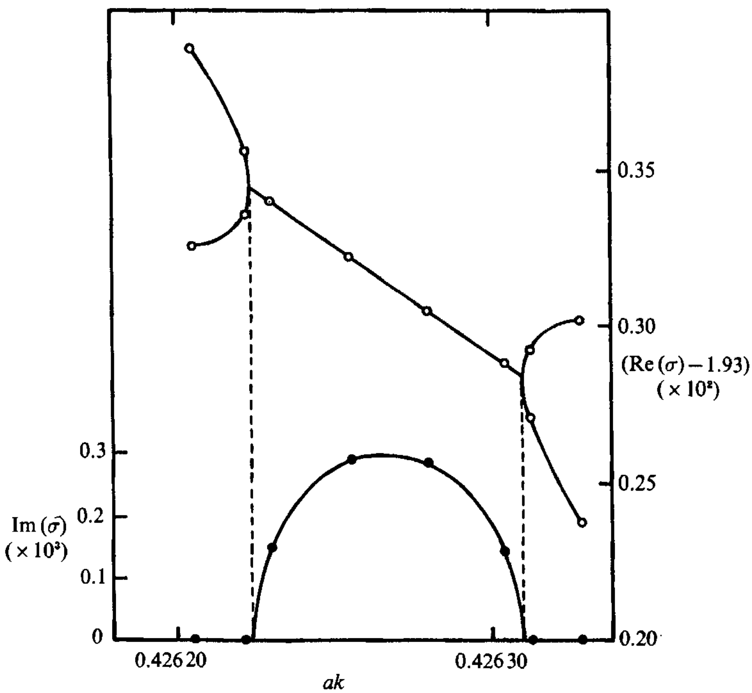

FIGURE 2. 'Bubble' of instability due to collision of the superharmonic modes $p+m = 5$ and $p-m = 1$ near $ak = 0.426$.

of 3×10^{-4} and agrees quite well with the value $(ak/\pi)^4 = 3\times 10^{-4}$ suggested by Zakharov (1968) for weakly resonating wave interactions.

For class II ($m = 2$) with ak close to 0.35 it appears that the instability zone crosses the point $p = 6$ on the p-axis. As shown in figure 3 this leads to a 'bubble of instability' due to the collision of the modes $p+m = 6+2$ and $p-m-1 = 6-2-1$

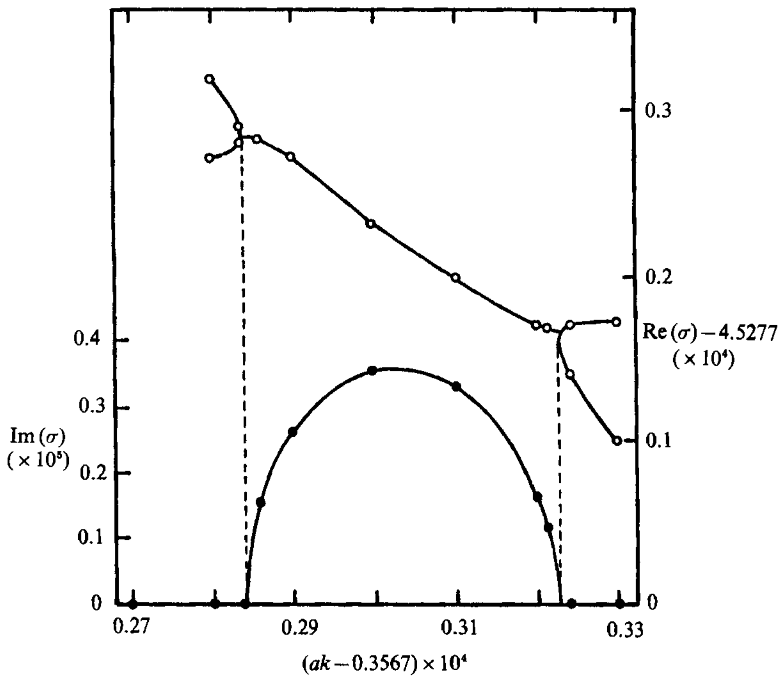

FIGURE 3. 'Bubble' of instability due to collision of the superharmonic modes $p+m = 8$ and $p-m = 3$ near $ak = 0.356$.

of opposite signatures. A truncation order of 70 is sufficient to obtain relevant eigenvalues and convergence as the wave steepness is now moderate. No appeal to Tanaka's technique was necessary; the computational procedure was similar to that used by Longuet-Higgins (1978*a*). It is seen that the maximum rate of growth is of the order of 0.4×10^{-5}. This is smaller than the expected value of $(ak/\pi)^5 = 2 \times 10^{-5}$ for weakly resonating interactions. No reason for this can be proposed at this time. An identical situation was reported by Hogan (1988) in his study of the superharmonic instabilities of capillary waves. From figure 1 a 'bubble of instability' of class II ($m = 2$) would appear for ak close to 0.43 as the instability region crosses the p-axis at $p = 5$, due to the collision of the modes $p+m = 5+2$ and $p-m-1 = 5-2-1$ of opposite signatures.

As mentioned previously, superharmonic perturbations do not generally correspond to the maximum rates of growth. Nevertheless, as will be reported in a subsequent article, they may appear together with the most unstable (subharmonic) perturbations under some circumstances.

4. Concluding remarks

The computations on the stability of gravity waves on deep water now cover almost the full range of admissible steepness. They are often tedious and costly especially when the steepness is large. Then a stability criterion, when available, is of considerable importance. This article illustrates the efficiency of the condition stated recently by MacKay & Saffman (1986). A stability prediction *appears possible* once the dispersion relation is known (see e.g. Voronovich, Lobanov & Rybak 1980). Examples in the field of wave–current interactions were reported by Kharif (1990).

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REFERENCES

BENJAMIN, T. B. & FEIR, J. E. 1967 The disintegration of wave trains on deep water. *J. Fluid Mech.* **27**, 417–430.

BRANGER, H., RAMAMONJIARISOA, A. & KHARIF, C. 1986 Sur la stabilité des vagues de gravité de fortes cambrures aux perturbations sous harmoniques. *J. Mec. Theor. Appl.* **5**, 535–550.

DYSTHE, K. B. 1979 Note on a modification to the non linear Schrödinger equation for application to deep water waves. *Proc. R. Soc. Lond.* A **369**, 105–114.

FEIR, J. E. 1967 Discussion: some results from wave pulse experiments. *Proc. R. Soc. Lond.* A **299**, 54.

HOGAN, S. J. 1988 The superharmonic normal mode instabilities of nonlinear deep-water capillary waves. *J. Fluid Mech.* **190**, 165–177.

KHARIF, C. 1987 Etude comparative des instabilités tridimensionnelles et bidimensionnelles des vagues de gravité de fortes cambrures. *J. Mec. Theo. Appl.* **6**, 843–864.

KHARIF, C. 1990 Some aspects of the kinematics of short waves over long waves. In *Water Wave kinematics Advanced Research Workshop* (ed. A. Tørum & O. T. Gudmestad). Kluwer.

KHARIF, C. & RAMAMONJIARISOA, A. 1988 Deep water gravity wave instabilities at large steepness. *Phys. Fluids* **31**, 1286–1288.

LAKE, B. M., YUEN, H. C., RUNGALDIER, H. & FERGUSON, W. E. 1977 Nonlinear deep-water waves: theory and experiments. Part 2. Evolution of a continuous wave train. *J. Fluid Mech.* **83**, 49–74.

LONGUET-HIGGINS, M. S. 1978*a* The instabilities of gravity waves of finite amplitude in deep-water. I. Super harmonics. *Proc. R. Soc. Lond.* A **360**, 471–488.

LONGUET-HIGGINS, M. S. 1978*b* The instabilities of gravity waves of finite amplitude in deep-water. II. Subharmonics. *Proc. R. Soc. Lond.* A **360**, 489–505.

LONGUET-HIGGINS, M. S. 1985 Bifurcation in gravity waves. *J. Fluid Mech.* **151**, 457–475.

LONGUET-HIGGINS, M. S. 1986 Bifurcation and instability in gravity waves. *Proc. R. Soc. Lond.* A **403**, 167–187.

MACKAY, R. S. & SAFFMAN, P. G. 1986 Stability of water waves. *Proc. R. Soc. Lond.* A **406**, 115–125.

MCLEAN, J. W. 1982 Instabilities of finite amplitude water waves. *J. Fluid Mech.* **114**, 315–330.

MCLEAN, J. W., MA, Y. C., MARTIN, D. V., SAFFMAN, P. G. & YUEN, H. C. 1981 Three-dimensional instability of finite amplitude water waves. *Phys. Rev. Lett.* **46**, 817–820.

STIASSNIE, M. & SHEMER, L. 1984 On modifications of the Zakharov equation for surface gravity waves. *J. Fluid Mech.* **143**, 47–67.

SU, M. Y. 1982 Three-dimensional deep-water waves. Part 1. Experimental measurement of skew and symmetric wave patterns. *J. Fluid Mech.* **124**, 73–108.

TANAKA, M. 1983 The stability of steep gravity waves. *J. Phys. Soc. Japan* **52**, 3047–3055.

TANAKA, M. 1985 The stability of steep gravity waves. Part 2. *J. Fluid Mech.* **156**, 281–289.

VORONOVICH, A. G., LOBANOV, E. D. & RYBAK, S. A. 1980 On the stability of gravitational–capillary waves in the presence of a vertically nonuniform current. *Izv. Atm. Ocean Phys.*, **16**, 220–222.

WITHAM, G. B. 1974 *Linear and Nonlinear Waves.* Wiley.

ZAKHAROV, V. E. 1968 Stability of period waves of finite amplitude on the surface of a deep fluid. *Z. Angew. Math. Phys.* **9**, 86–94.

ZHANG, J. & MELVILLE, W. K. 1987 Three-dimensional instabilities of nonlinear gravity–capillary waves. *J. Fluid Mech.* **174**, 187–208.